

Classical Real Time Correlation Functions And Quantum Corrections at Finite Temperature

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Abstract

We consider quantum corrections to classical real time correlation functions at finite temperature. We derive a semi-classical expansion in powers of \hbar with coefficients including all orders in the coupling constant. We give explicit expressions up to order \hbar^2 . We restrict ourselves to a scalar theory. This method, if extended to gauge theories, might be used to compute quantum corrections to the high temperature baryon number violation rate in the Standard Model.

I. INTRODUCTION

In the Standard Model baryon number is not conserved [1]. While the rate for baryon number violating processes is exponentially suppressed at zero temperature it can be significant at temperatures of the order of the electroweak scale [2–7]. This could have had an influence on the baryon asymmetry of the universe (for a recent review see [8]).

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Baryon number nonconserving processes are mediated by topologically nontrivial vacuum-to-vacuum transitions in the electroweak theory. These transitions are characterized by a change $\Delta N_{\text{CS}} = \pm 1$ of the Chern–Simons number

$$N_{\text{CS}} = \frac{g^2}{32\pi^2} \epsilon_{ijk} \int d^3x \left(F_{ij}^a A_k^a - \frac{g}{3} \epsilon^{abc} A_i^a A_j^b A_k^c \right) \quad (1)$$

where A_μ^a are the SU(2) gauge fields and g is the weak coupling constant. The change of baryon number ΔB is related to ΔN_{CS} by

$$\Delta B = n_f \Delta N_{\text{CS}} \quad (2)$$

where n_f is the number of fermion families. At finite temperature the N_{CS} –changing transitions occur by thermal fluctuations across an energy barrier. For temperatures less than the critical temperature of the electroweak phase transition $T_c \sim 100$ GeV the transition rate can be calculated in a weak coupling expansion around the so called sphaleron which is a saddle point on this barrier.

The sphaleron approximation can not be extended to temperatures above T_c . The field configurations which dominate the rate for $T \gg T_c$ have a typical size of order $\sim 1/(g^2 T)$ [6]. This is the scale at which perturbation theory at finite temperature breaks down [9]. For this temperature range the baryon number violation rate per unit volume has been estimated as [6]

$$\Gamma = \kappa (\alpha_W T)^4 \quad (3)$$

where κ is a dimensionless constant and $\alpha_W = g^2/(4\pi)$.

Grigoriev and Rubakov [10] have suggested a numerical algorithm for computing Γ . They argued that the thermal activation across a barrier is a classically allowed process and that one should determine the rate in a purely classical field theory. This consists of solving the classical equations of motion for the gauge fields and measuring the change of the Chern–Simons number. Finally the average over initial conditions has to be performed with the Boltzmann weight $\exp(-\beta H)$. Since the relevant field modes are long wavelength and involve many quanta one would expect that they in fact behave classically.

The quantity of interest is the real time correlation function

$$C_{\text{CS}}(t) = \langle [N_{\text{CS}}(t) - N_{\text{CS}}(0)]^2 \rangle \quad (4)$$

where $\langle \dots \rangle$ denotes the average over a thermal ensemble. It is commonly assumed that the Chern–Simons number performs a random walk so that for large times $C_{\text{CS}}(t)$ grows linearly with t . Then the rate Γ is related to $C_{\text{CS}}(t)$ through

$$C_{\text{CS}}(t) \rightarrow V\Gamma t \quad (5)$$

for $t \rightarrow \infty$ where V is the space volume [7]. Using real time evolution of classical equations of motion on a Hamiltonian lattice Ambjørn et al. [11] obtained a lower bound for Γ . Ambjørn and Krasnitz [12] performed a more precise calculation for the pure SU(2) gauge theory, they found $\Gamma = 1.1(\alpha_W T)^4$. This result has been confirmed by Moore [13,14] who considered the change of N_{CS} in the presence of a chemical potential for N_{CS} . Tang and Smit [15] have used the algorithm of Ref. [12] to study the the SU(2)–Higgs model. For $T \gg T_c$ their results agree with those of Ref. [12]. For temperatures less than T_c they found a result which is much larger than the one obtained in the sphaleron approximation [5,6]. Moore and Turok [16] found similar results for $T < T_c$ but they argue that for this temperature range the classical approximation for the correlation function might not be reliable.

There are several questions arising in this context. It has been known since the work of Rayleigh, Jeans, Planck and Einstein that statistical mechanics for a field theory is ill defined due to ultraviolet divergences. Many physical quantities depend on the cutoff and do not have a finite continuum limit when computed classically. For static quantities the classical theory is equivalent to a three dimensional Euclidean quantum field theory [17–21]. It is possible to introduce counterterms in the classical Hamiltonian such that the continuum limit exists. Since the three dimensional theory is super–renormalizable only a finite number of divergences occur. For time dependent quantities the situation is more complicated. In gauge theories there are additional ultraviolet divergences not related to the three dimensional theory [22]. These arise in diagrams which in the corresponding quantum theory give rise to

terms $\propto T^2$, the so called hard thermal loops [23]. These are explicitly time dependent. The hard thermal loops are due to large loop momenta $k \sim T$ which are much larger than the momentum scale $\sim g^2 T$ associated with the N_{CS} -diffusion. If the high momentum modes decouple the continuum limit of certain quantities may exist. The results of Refs. [12–16] indicate that this is the case for Γ in Eq. (5) when computed for temperatures much larger than T_c .

If the continuum limit of the classical correlation function for some particular quantity exists, the question remains how accurate the classical approximation is. In Ref. [16] this problem has been addressed perturbatively. It would be desirable to find a way to go beyond perturbation theory and to find an algorithm for incorporating quantum corrections into the classical lattice calculations done so far.

In the present paper we address the latter problem. We develop a semi-classical expansion of real time correlation functions at finite temperature starting from the quantum mechanical correlator. We keep Planck's constant $\hbar \neq 1$. We derive an expansion in powers of \hbar . In the limit $\hbar \rightarrow 0$ we recover the classical limit. We consider correlation functions of the type

$$C(t) = \left\langle \frac{1}{2} [A(0)B(t) + B(t)A(0)] \right\rangle \quad (6)$$

where $A(t)$ and $B(t)$ are functions of the Heisenberg field operators at time t . The operator ordering in Eq. (6) is the relevant one for the Chern–Simons number diffusion. The problem of ultraviolet divergences related to real time evolution will not be addressed here.

The semiclassical expansion for static finite temperature correlation functions has been developed in the early days of quantum mechanics [24] and can be found in textbooks, e.g., Ref. [25]. The author is not aware of such an expansion for time dependent quantities existing in the literature.

In Section II we illustrate the method for the simplest quantum mechanical system, i.e., for one degree of freedom. The generalization to a system with an arbitrary number of degrees of freedom (Sect. III) and thus to field theory (Sect. IV) is straightforward. Finally we summarize and discuss the results in Sect. V.

II. QUANTUM MECHANICS WITH ONE DEGREE OF FREEDOM

We consider one quantum mechanical degree of freedom q with conjugate momentum p and the Hamiltonian $H = p^2/2 + U(q)$. The potential U is assumed to be a fourth order polynomial in q . The quantum operators are denoted by capital letters. Heisenberg and Schrödinger operators are denoted by $Q(t)$, $P(t)$ and $Q = Q(0)$, $P = P(0)$, respectively. We consider the two point function

$$C(t) = Z^{-1} \text{tr} \left\{ \frac{1}{2} [Q(t)Q(0) + Q(0)Q(t)] e^{-\beta H(P,Q)} \right\} \quad (7)$$

where $Z = \text{tr} \{ \exp(-\beta H) \}$. We will find an expansion

$$C(t) = C_0(t) + \hbar C_1(t) + \hbar^2 C_2(t) + \dots \quad (8)$$

Following [25] we use (Schrödinger) momentum eigenstates $|p\rangle$ to perform the trace and insert $1 = \int dq |q\rangle \langle q|$ to obtain

$$C(t) = Z^{-1} \int \frac{dp dq}{2\pi\hbar} I \quad (9)$$

with

$$I = \langle q|p\rangle \langle p| \frac{1}{2} [Q(t)Q(0) + Q(0)Q(t)] e^{-\beta H(P,Q)} |q\rangle. \quad (10)$$

Factorizing the Boltzmann factor one writes I in terms of χ as

$$I = \exp\{-\beta H(p, q)\} \chi \quad (11)$$

Differentiating Eq. (10) with respect to β one obtains [25]

$$\begin{aligned} \partial_\beta \chi &= -i\hbar p \left[-\beta U'(q) \chi + \partial_q \chi \right] \\ &+ \frac{1}{2} \hbar^2 \left[\partial_q^2 \chi - 2\beta U'(q) \partial_q \chi - \beta U''(q) \chi + \beta^2 U'(q)^2 \chi \right] \end{aligned} \quad (12)$$

where the primes denote derivatives with respect to q . Eq. (12) can be solved by iteration when χ is expanded in powers of \hbar ,

$$\chi = \chi_0 + \hbar\chi_1 + \hbar^2\chi_2 + \dots \quad (13)$$

We have to determine the boundary condition at $\beta = 0$,

$$\chi^0 \equiv \chi(\beta = 0) = \langle q|p\rangle\langle p|\frac{1}{2}(Q(t)Q(0) + Q(0)Q(t))|q\rangle, \quad (14)$$

which will be expanded in powers of \hbar as well:

$$\chi^0 = \chi_0^0 + \hbar\chi_1^0 + \hbar^2\chi_2^0 + \dots \quad (15)$$

We write χ^0 as

$$\chi^0 = \left(q + \frac{i}{2}\hbar\partial_p\right)\eta \quad (16)$$

with

$$\eta \equiv \langle q|p\rangle\langle p|Q(t)|q\rangle. \quad (17)$$

Differentiating with respect to t we find

$$\partial_t\eta = \frac{i}{\hbar}\langle q|p\rangle\langle p|[H(P, Q), Q(t)]|q\rangle \quad (18)$$

where we have used the fact that the Hamilton operator is time-independent. From Eq. (18) we obtain

$$\begin{aligned} \partial_t\eta &= \frac{i}{\hbar}\left(-i\hbar p + \frac{i}{2}\hbar^2\partial_q^2 + U(q + i\hbar\partial_p) - U(q)\right)\eta \\ &= p\partial_q\eta - U'(q)\partial_p\eta + \frac{i}{2}\hbar\left(\partial_q^2\eta - U''(q)\partial_p^2\eta\right) + \frac{1}{6}\hbar^2U'''(q)\partial_p^3\eta + \frac{i}{24}\hbar^3U^{(4)}\partial_p^4\eta. \end{aligned} \quad (19)$$

The initial condition for η is

$$\eta(t = 0) = q. \quad (20)$$

We expand η in powers of \hbar

$$\eta = \eta_0 + \hbar\eta_1 + \hbar^2\eta_2 + \dots \quad (21)$$

and solve Eq. (19) by iteration. For η_0 we obtain the differential equation

$$\begin{aligned}\partial_t \eta_0 &= (p \partial_q - U'(q) \partial_p) \eta_0 \\ &= \{H, \eta_0\}\end{aligned}\tag{22}$$

where $\{, \}$ denotes the Poisson bracket

$$\{f, g\} = \partial_p f \partial_q g - \partial_p g \partial_q f\tag{23}$$

The general solution to Eq. (22) is $\eta_0 = f(p_c(t), q_c(t))$ where $p_c(t)$, $q_c(t)$ are the solutions to the classical Hamilton equations of motion

$$\dot{q}_c(t) = \{H, q_c(t)\}, \quad \dot{p}_c(t) = \{H, p_c(t)\}\tag{24}$$

with the initial conditions

$$q_c(0) = q, \quad p_c(0) = p.\tag{25}$$

With Eq. (20) we obtain

$$\eta_0 = q_c(t)\tag{26}$$

and therefore $\chi_0^0 = qq_c(t)$. From Eq. (12) we see that χ_0 is independent of β . Thus

$$\chi_0 = qq_c(t).\tag{27}$$

Then we finally obtain

$$C_0(t) = Z_0^{-1} \int \frac{dp dq}{2\pi \hbar} e^{-\beta H(p, q)} qq_c(t)\tag{28}$$

with $Z_0 = \int \frac{dp dq}{2\pi \hbar} e^{-\beta H(p, q)}$, which corresponds to the Grigoriev-Rubakov expression.

Now we consider the order \hbar correction to Eq. (28). From Eq. (12) we obtain

$$\chi_1 = \chi_1^0 - ip \left(\beta \chi_0' - \frac{1}{2} \beta^2 U' \chi_0 \right).\tag{29}$$

In the correlator $C(t)$ the function χ_1 only appears in the form $\int dp dq \exp(-\beta H) \chi_1$. One can integrate this by parts to find the equivalent expression

$$\int dp dq e^{-\beta H(p,q)} \chi_1 = \int dp dq e^{-\beta H(p,q)} \left\{ \chi_1^0 - \frac{i}{2} \partial_p \partial_q \chi_0 \right\}. \quad (30)$$

Eq. (16) gives

$$\chi_1^0 = q \eta_1 + \frac{i}{2} \partial_p \eta_0 \quad (31)$$

where η_1 is determined by

$$\partial_t \eta_1 = \{H, \eta_1\} + \frac{i}{2} (\partial_q^2 - U''(q) \partial_p^2) \eta_0 \quad (32)$$

with the initial condition $\eta_1(t=0) = 0$. In order to solve Eq. (32) it is convenient to rewrite it as

$$\partial_t \left(\eta_1 - \frac{i}{2} \partial_p \partial_q \eta_0 \right) = \{H, \eta_1 - \frac{i}{2} \partial_p \partial_q \eta_0\}. \quad (33)$$

Therefore

$$\eta_1 = \frac{i}{2} \partial_p \partial_q q_c(t). \quad (34)$$

Combining Eqs. (31) and (34) we arrive at

$$\chi_1^0 = \frac{i}{2} \partial_p \partial_q \chi_0. \quad (35)$$

If now compare Eqs. (30) and (35) we see that

$$C_1(t) = 0, \quad (36)$$

i.e. the order \hbar correction to the classical correlation function (28) vanishes.

Next we compute the $\mathcal{O}(\hbar^2)$ corrections to C_0 . The β -dependent part of χ_2 is determined by Eq. (12). Using the results for χ_1 (Eqs. (29) and (35)) we obtain the lengthy expression

$$\begin{aligned} \chi_2 = & \chi_2^0 + \frac{1}{2} \left(\beta \partial_q^2 \chi_0 - \beta^2 U'(q) \partial_q \chi_0 - \frac{1}{2} \beta^2 U''(q) \chi_0 + \frac{1}{3} \beta^3 (U'(q))^2 \chi_0 \right) \\ & + p \left(\frac{1}{2} \beta \partial_p \partial_q^2 \chi_0 - \frac{1}{4} \beta^2 U'(q) \partial_p \partial_q \chi_0 \right) \\ & + p^2 \left(-\frac{1}{2} \beta^2 \partial_q^2 \chi_0 + \frac{1}{6} \beta^3 U''(q) \chi_0 + \frac{1}{2} \beta^3 U'(q) \partial_q \chi_0 - \frac{1}{8} \beta^4 (U'(q))^2 \chi_0 \right). \end{aligned} \quad (37)$$

It can, however, be simplified by partial integrations:

$$\begin{aligned} \int dp dq e^{-\beta H(p,q)} \chi_2 &= \int dp dq e^{-\beta H(p,q)} \left\{ \chi_2^0 + \frac{1}{8} \partial_p^2 \partial_q^2 \chi_0 \right. \\ &\quad \left. + \frac{\beta}{24} [\partial_q^2 \chi_0 + U''(q) \partial_p^2 \chi_0 - \beta U'''(q) \chi_0] \right\}. \end{aligned} \quad (38)$$

We use Eq. (16) to write

$$\chi_2^0 = q \eta_2 + \frac{i}{2} \partial_p \eta_1 \quad (39)$$

where η_2 satisfies

$$\partial_t \eta_2 = \{H, \eta_2\} + \frac{i}{2} (\partial_q^2 - U''(q) \partial_p^2) \eta_1 + \frac{1}{6} U'''(q) \partial_p^3 \eta_0 \quad (40)$$

with the initial condition $\eta_2(t=0) = 0$.

To compute η_2 we rewrite Eq. (40) as

$$\partial_t \left(\eta_2 + \frac{1}{8} \partial_p^2 \partial_q^2 \eta_0 \right) = \{H, \eta_2 + \frac{1}{8} \partial_p^2 \partial_q^2 \eta_0\} + \frac{1}{24} U'''(q) \partial_p^3 \eta_0. \quad (41)$$

Thus

$$\eta_2 = -\frac{1}{8} \partial_p^2 \partial_q^2 \eta_0 + \bar{\eta}_2 \quad (42)$$

where $\bar{\eta}_2$ satisfies

$$\partial_t \bar{\eta}_2 = \{H, \bar{\eta}_2\} - \frac{1}{24} U'''(q) \{q, q_c(t)\}_3. \quad (43)$$

Here we have denoted

$$\{f, g\}_0 = g, \quad \{f, g\}_{n+1} = \{f, \{f, g\}_n\} \quad (44)$$

in order to write the derivatives ∂_p in terms of Poisson brackets. With the initial condition $\bar{\eta}_2(t=0)$ we find

$$\bar{\eta}_2 = -\frac{1}{24} \int_0^t dt' U'''(q_c(t')) \{q_c(t'), q_c(t)\}_3. \quad (45)$$

Here we have used the fact that the Poisson bracket is invariant under canonical transformations of the variables p and q and that the time evolution can be viewed as a canonical transformation.

Putting everything together we can write the correlation function (7) as

$$C(t) = Z^{-1} \int \frac{dp dq}{2\pi\hbar} e^{-\beta H(p,q)} \left\{ \left[1 + \frac{\hbar^2 \beta}{24} \left(\partial_q^2 + U''(q) \partial_p^2 - \beta U'''(q) \right) \right] q q_c(t) + \hbar^2 q \bar{\eta}_2 \right\} + \mathcal{O}(\hbar^3) \quad (46)$$

where $\bar{\eta}_2$ is given by Eq. (45).

III. MANY DEGREES OF FREEDOM

The results obtained in the previous section can be easily generalized to a system with an arbitrary number of bosonic degrees of freedom. The coordinates q_i and momenta p_i will be collectively denoted by q and p . The notation is completely analogous to the one in Sect. II. The Hamiltonian is $H = 1/2 \sum_i p_i^2 + U(q)$. We consider the correlator

$$C(t) = Z^{-1} \text{tr} \left\{ \frac{1}{2} [A(Q)B(Q(t)) + B(Q(t))A(Q)] e^{-\beta H(P,Q)} \right\} \quad (47)$$

where $A(q)$ and $B(q)$ are some functions of q . We write $C(t)$ as

$$C(t) = Z^{-1} \int \prod_i \left(\frac{dp_i dq_i}{2\pi\hbar} \right) e^{-\beta H(p,q)} \chi \quad (48)$$

The differential equation determining the β dependence of χ is [25]

$$\begin{aligned} \partial_\beta \chi = \sum_i \left\{ -i\hbar p_i (\partial_{q_i} \chi - \beta U_i(q) \chi) \right. \\ \left. + \frac{\hbar^2}{2} \left(\partial_{q_i}^2 \chi - 2\beta U_i(q) \partial_{q_i} \chi - \beta U_{ii}(q) \chi + \beta^2 (U_i(q))^2 \chi \right) \right\} \end{aligned} \quad (49)$$

where a subscript i on U denotes the derivative ∂_{q_i} .

The boundary condition for Eq. (49) is $\chi(\beta = 0) = \chi^0$ with

$$\chi^0 = \frac{1}{2} \langle q|p \rangle \langle p| [A(Q)B(Q(t)) + B(Q(t))A(Q)] |q \rangle. \quad (50)$$

This can be written as

$$\chi^0 = \frac{1}{2}[A(q) + A(q + i\hbar\partial_p)]\eta \quad (51)$$

with

$$\eta = \langle q|p\rangle\langle p|B(Q(t))|q\rangle. \quad (52)$$

The time dependence of η is determined by

$$\begin{aligned} \partial_t \eta = & \sum_i (p_i \partial_{q_i} \eta - U_i(q) \partial_{p_i} \eta) + \frac{i}{2} \hbar \left[\sum_i \partial_{q_i}^2 \eta - \sum_{ij} U_{ij}(q) \partial_{p_i} \partial_{p_j} \eta \right] \\ & + \frac{1}{6} \hbar^2 \sum_{ijk} U_{ijk}(q) \partial_{p_i} \partial_{p_j} \partial_{p_k} \eta + \frac{i}{24} \hbar^3 \sum_{ijkl} U_{ijkl}(q) \partial_{p_i} \partial_{p_j} \partial_{p_k} \partial_{p_l} \eta. \end{aligned} \quad (53)$$

Note that that this differential equation is completely analogous to Eq. (19). It does not involve the function B which enters only through the initial condition

$$\eta(t=0) = B(q). \quad (54)$$

Again we expand $\eta = \eta_0 + \hbar\eta_1 + \hbar^2\eta_2 + \dots$ and solve Eq. (53) by iteration. For the first term in this expansion we obtain

$$\eta_0 = B(q_c(t)) \quad (55)$$

where $q_c(t)$ is the solution to the classical equations of motion with initial condition $q_c(0) = q$, $\dot{q}_c(0) = p$. The $\mathcal{O}(\hbar)$ correction to Eq. (55) is given through

$$\eta_1 = \frac{i}{2} \sum_i \partial_{q_i} \partial_{p_i} \eta_0 \quad (56)$$

and the result for η_2 is

$$\eta_2 = -\frac{1}{8} \sum_{ij} \partial_{q_i} \partial_{p_i} \partial_{q_j} \partial_{p_j} \eta_0 + \bar{\eta}_2 \quad (57)$$

where

$$\bar{\eta}_2 = -\frac{1}{24} \int_0^t dt' U_{ijk}(q_c(t')) \{q_{ci}(t'), \{q_{cj}(t'), \{q_{ck}(t'), B(q_c(t))\}\}\}. \quad (58)$$

Now we can write down the first three terms of the expansion (15) for χ^0 :

$$\chi_0^0 = A(q)B(q_c(t)) \quad (59)$$

$$\chi_1^0 = \frac{i}{2} \sum_i \partial_{q_i} \partial_{p_i} \chi_0^0 \quad (60)$$

and

$$\chi_2^0 = -\frac{1}{8} \sum_{ij} \left\{ \partial_{q_i} \partial_{p_i} \partial_{q_j} \partial_{p_j} \chi_0^0 + A_{ij}(q) \partial_{p_i} \partial_{p_j} B(q_c(t)) \right\} + A(q) \bar{\eta}_2 \quad (61)$$

with $A_{ij} = \partial_{q_i} \partial_{q_j} A$.

Now that we know the boundary condition at $\beta = 0$ it is straightforward to solve Eq. (49) to determine χ . The result is somewhat lengthy but it simplifies after integrating $\int \prod_i (dp_i dq_i) / (2\pi\hbar) \exp(-\beta H) \chi$ by parts. As in Sect. II one finds that the $\mathcal{O}(\hbar)$ contribution to $C(t)$ vanishes and we obtain

$$\begin{aligned} C(t) = & Z^{-1} \int \prod_i \left(\frac{dp_i dq_i}{2\pi\hbar} \right) e^{-\beta H(p,q)} \\ & \times \left\{ \left[1 + \frac{\hbar^2 \beta}{24} \left(\sum_i \partial_{q_i}^2 + \sum_{ij} U_{ij}(q) \partial_{p_i} \partial_{p_j} - \beta \sum_i U_{ii}(q) \right) \right] A(q) B(q_c(t)) \right. \\ & \left. + \hbar^2 \left(A(q) \bar{\eta}_2 - \frac{1}{8} \sum_{ij} A_{ij}(q) \partial_{p_i} \partial_{p_j} B(q_c(t)) \right) \right\} \\ & + \mathcal{O}(\hbar^3). \end{aligned} \quad (62)$$

IV. FIELD THEORY

The field theoretical expressions for the correlation function are completely analogous to the ones found in the previous section. We consider a theory with one scalar field φ and its conjugate momentum π . We consider the correlation function

$$C(t) = Z^{-1} \text{tr} \left\{ \frac{1}{2} \left[\Phi(0, \vec{x}_A) \Phi(t, \vec{x}_B) + \Phi(t, \vec{x}_B) \Phi(0, \vec{x}_A) \right] e^{-\beta H[\Pi, \Phi]} \right\} \quad (63)$$

for the Hamiltonian

$$H[\pi, \varphi] = \int d^3x \frac{1}{2} \pi^2(\vec{x}) + U[\varphi] \quad (64)$$

where

$$U[\varphi] = \int d^3x \left\{ \frac{1}{2} (\nabla \varphi(\vec{x}))^2 + V(\varphi(\vec{x})) \right\}. \quad (65)$$

Replacing

$$\sum_i \partial_{p_i} \partial_{q_i} \rightarrow \int d^3x \frac{\delta}{\delta \pi(\vec{x})} \frac{\delta}{\delta \varphi(\vec{x})} \quad (66)$$

the \hbar expansion of (63) can be read off from Eqs. (58) and (62):

$$\begin{aligned} C(t) = Z^{-1} \int [d\pi d\varphi] e^{-\beta H[\pi, \varphi]} & \left\{ \left[1 + \frac{\hbar^2 \beta}{24} \int d^3x \left(\frac{\delta^2}{\delta \varphi(\vec{x})^2} \right. \right. \right. \\ & + \int d^3x' \delta(\vec{x} - \vec{x}') \left(-\nabla_x^2 + V''(\varphi(\vec{x})) \right) \frac{\delta^2}{\delta \pi(\vec{x}) \delta \pi(\vec{x}')} \\ & \left. \left. \left. - \beta \frac{\delta^2 U[\varphi]}{\delta \varphi(\vec{x})^2} \right) \right] \varphi(\vec{x}_A) \varphi_c(t, \vec{x}_B) + \hbar^2 \varphi(\vec{x}_A) \bar{\eta}_2 \right\} + \mathcal{O}(\hbar^3) \end{aligned} \quad (67)$$

where φ_c is the classical field satisfying the equation of motion

$$\ddot{\varphi}_c - \nabla^2 \varphi_c + V'(\varphi_c) = 0 \quad (68)$$

with the initial conditions

$$\varphi_c(0, \vec{x}) = \varphi(\vec{x}), \quad \dot{\varphi}_c(0, \vec{x}) = \pi(\vec{x}). \quad (69)$$

Furthermore, $\bar{\eta}_2$ in Eq. (67) is given by

$$\bar{\eta}_2 = -\frac{1}{24} \int_0^t dt' \int d^3x V'''(\varphi_c(t', \vec{x})) \{ \varphi_c(t', \vec{x}), \varphi_c(t, \vec{x}_B) \}_3 \quad (70)$$

where $\{, \}_3$ is defined as in Eq. (44) and the Poisson brackets are

$$\{f, g\} = \int d^3x \left(\frac{\delta f}{\delta \pi(\vec{x})} \frac{\delta g}{\delta \varphi(\vec{x})} - \frac{\delta g}{\delta \pi(\vec{x})} \frac{\delta f}{\delta \varphi(\vec{x})} \right). \quad (71)$$

The right hand side of Eq. (67) contains ultraviolet divergences. Consider, e.g., the term $\delta^2 U[\varphi] / \delta \varphi(\vec{x})^2$. The singularities arise because we take two functional derivatives of U with

respect to $\varphi(\vec{x})$, both of them for the same space point \vec{x} . If we write $\delta^2 U[\varphi]/\delta\varphi(\vec{x})^2$ in momentum space and introduce a momentum cutoff Λ it becomes

$$\frac{\delta^2 U[\varphi]}{\delta\varphi(\vec{x})^2} = \frac{1}{2\pi^2} \left(\frac{1}{5} \Lambda^5 + \frac{1}{3} \Lambda^3 V''(\varphi(\vec{x})) \right). \quad (72)$$

The parameters in the Hamiltonian (64) are bare quantities which have to be expressed in terms of renormalized ones. In the following we will see how the singularities are canceled after renormalization in the limit $t \rightarrow 0$ of $C(t)$. In this limit the problem is of course much simpler. Ultraviolet divergences related to real time evolution have been discussed in Ref. [22] and some of them turned out to be rather nontrivial even in the classical limit i.e. without including the $\mathcal{O}(\hbar^2)$ corrections. As it was already mentioned the numerical calculations done so far indicate that nevertheless the continuum limit of the classical $\mathcal{O}(\hbar^0)$ result for Γ (Eq. (5)) exists. Therefore it might even be meaningful to consider quantum corrections leaving aside ultraviolet problems for a moment.

For definiteness we choose the potential in (65) as

$$V(\varphi) = \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4. \quad (73)$$

For $t = 0$ the path integral over π is Gaussian and π can be integrated out. This gives a constant factor which is canceled by the corresponding factor in Z and which will be ignored in the following. Eq. (67) becomes

$$\begin{aligned} C(0) = Z^{-1} \int [d\varphi] e^{-\beta U[\varphi]} \\ \times \left\{ \left[1 - \frac{\hbar^2 \beta^2}{24} \frac{1}{2\pi^2} \int d^3x \left(\frac{1}{5} \Lambda^5 + \frac{1}{3} \Lambda^3 \left(m^2 + \frac{\lambda}{2} \varphi^2(\vec{x}) \right) \right) \right] \varphi(\vec{x}_A) \varphi(\vec{x}_B) \right. \\ \left. + \frac{\hbar^2 \beta}{12} \delta(\vec{x}_A - \vec{x}_B) \right\} + \mathcal{O}(\hbar^3). \end{aligned} \quad (74)$$

We have to express the bare parameters in terms of the renormalized ones which we denote by a subscript r. We write

$$U = U_r + \delta U. \quad (75)$$

We include a field independent part in the Hamiltonian which we denote by U^0 . The counterterm δU^0 is written as a power series in \hbar , $\delta U^0 = \delta U_0^0 + \hbar^2 \delta U_2^0 + \dots$. Then one has to choose

$$\delta U_2^0 = -\frac{\beta}{24} \frac{1}{2\pi^2} \int d^3x \left(\frac{1}{5} \Lambda^5 + \frac{1}{3} \Lambda^3 m^2 \right) \quad (76)$$

to cancel the first two singular terms in Eq. (74).

For the mass renormalization we write

$$m_{\text{r}}^2 = m^2 - \delta m^2. \quad (77)$$

We define the renormalized mass in the imaginary time formalism (see, e.g., [26]):

$$m_{\text{r}}^2 = m^2 + \Pi(\omega_n = 0, \mu^2) \quad (78)$$

where $\Pi(\omega_n, \vec{k}^2)$ is the self energy for Matsubara frequency $\omega_n = n2\pi T/\hbar$ and momentum \vec{k} and μ is some renormalization point. If μ is chosen sufficiently large, m_{r}^2 can be calculated in perturbation theory. The $\mathcal{O}(\lambda)$ contribution to δm^2 is given by the diagram (a) in Fig.1:

$$\delta m^2 = -\frac{\lambda}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} + \mathcal{O}(\lambda^2). \quad (79)$$

We can expand δm^2 in powers of \hbar ,

$$\delta m^2 = \delta m_0^2 + \hbar^2 \delta m_2^2 + \mathcal{O}(\hbar^4). \quad (80)$$

The zero Matsubara frequency part of Eq. (79) gives

$$\delta m_0^2 = -\frac{\lambda_{\text{r}}}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2 + m^2} + \mathcal{O}(\lambda^2). \quad (81)$$

The nonzero Matsubara frequency part can be expanded in powers of \hbar^2 . With a momentum cutoff Λ the first term in this series gives

$$\delta m_2^2 = -\frac{\lambda\beta}{24} \frac{1}{2\pi^2} \frac{\Lambda^3}{3}. \quad (82)$$

The zero Matsubara frequency parts of all higher loop diagrams for δm^2 (e.g. the diagram (b) in Fig. 1) contribute to δm_0^2 . However if one goes beyond $\mathcal{O}(\hbar^0)$ there are always at least

two propagators in which one particular nonzero Matsubara frequency appears. Therefore the \hbar expansion for these diagrams beyond $\mathcal{O}(\hbar^0)$ starts at $\mathcal{O}(\hbar^4)$ and Eq. (82) is valid to all orders in λ . For the coupling constant and wave function renormalization there is no $\mathcal{O}(\hbar^2)$ contribution even at the one loop level. In general there are finite counterterms of the order \hbar^0 which, however, do not have to be specified for the present purpose.

Replacing $U \rightarrow U_r + \delta U_0 + \hbar^2 \delta U_2$ in Eq. (74) and expanding in powers of \hbar the $\mathcal{O}(\hbar^2)$ divergences in Eq. (74) cancel and we obtain

$$C(0) = Z^{-1} \int [d\varphi] \exp \{ -\beta (U_r[\varphi] + \delta U_0[\varphi]) \} \\ \times \left\{ \varphi(\vec{x}_A) \varphi(\vec{x}_B) + \frac{\hbar^2 \beta}{12} \delta(\vec{x}_A - \vec{x}_B) \right\} + \mathcal{O}(\hbar^3) \quad (83)$$

This expression is similar to a dimensionally reduced theory [21]. There are two differences. First, the renormalized parameters in Eq. (83) are different from the ones in Ref. [21]. This difference is due to finite parts in the counterterm $\delta U_0[\varphi]$ corresponding to different renormalization conditions. The choice in Ref. [21] was made such that the Greens functions in the 3-dimensional theory and the Greens functions with $\omega_n = 0$ in the 4-dimensional theories agree up to a certain order in the coupling constant. Furthermore, in Ref. [21] the renormalized mass was written in terms of the physical Higgs mass.

The second difference between (83) and the expressions given in Ref. [21] is the appearance of the δ -function in Eq. (83). This term arises because in the formalism described in the present paper there is no separation of the fields into zero and nonzero Matsubara frequency components. Therefore, the correlation function (83) also contains the diagrams in which the external lines have nonzero Matsubara frequency. In momentum space the tree level propagator for the $\omega_n \neq 0$ fields is $D(\omega_n, \vec{k}^2) = 1/(\omega_n^2 + \vec{k}^2)$. Expanding this expression in powers of \hbar gives $\hbar^2/(2\pi nT)^2 + \mathcal{O}(\hbar^4)$. Summing over all $n \neq 0$ we obtain the δ -function-term in Eq. (83). All higher loop contributions to $D(\omega_n, \vec{k}^2)$ are $\mathcal{O}(\hbar^4)$ or higher. Therefore they do not contribute to $C(0)$ at the order considered here.

V. SUMMARY AND DISCUSSION

For real time correlation functions at finite temperature we have obtained an expansion in powers of Planck's constant \hbar which, at lowest order, corresponds to the classical correlation function in the sense of Grigoriev and Rubakov [10]. We have restricted ourselves to a scalar theory. For the correlation functions of the type (6) we found that the first order correction in \hbar vanishes. The first nonvanishing contribution is of the order \hbar^2 . The result has been written in terms of the solution to the classical field equations and its functional derivatives with respect to the initial conditions. The expansion outlined in this paper can be easily extended to higher orders in \hbar .

The method outlined in this paper might serve as a starting point for numerical computations of quantum corrections to classical correlation functions which can not be calculated in perturbation theory. An important example is the Chern-Simons number diffusion rate in the electroweak theory which determines the rate for baryon number violating processes in the early universe.

The $\mathcal{O}(\hbar^2)$ correction to the classical correlation function contains a term (70) which is nonlocal in time. One may ask how this term can be implemented in a numerical evaluation of the correlation function. It might be possible to find a set of local equations of motion such that their solution reproduces Eq. (70). In Eq. (67) the derivatives with respect to the initial conditions can be transformed into local operators at $t = 0$ through partial integrations of the path integral.

An important task is to extend the method outlined here to gauge theories. Another important issue are the ultraviolet divergences of the classical result and of the $\mathcal{O}(\hbar^2)$ corrections. For the pure gauge theory considered in Refs. [12,13] this maybe not so urgent since there the classical results seem to have a well behaved continuum limit. Then the ultraviolet behavior of the $\mathcal{O}(\hbar^2)$ corrections may not be that critical either. However, if the Higgs field is included as in Ref. [15] the problem of ultraviolet divergences is crucial.

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REFERENCES

- [1] G. t'Hooft, Phys. Rev. Lett. **37** (1976) 8; Phys. Rev. D **14** (1976) 3432.
- [2] A. D. Linde, Phys. Lett. **70B** (1977) 306.
- [3] S. Dimopoulos, L. Susskind, Phys. Rev. D **18** (1978) 4500.
- [4] N. Manton, Phys. Rev. D **28** (1983) 2019; F. Klinkhamer, N. Manton, Phys. Rev. D **30** (1984) 2212.
- [5] V. A. Kuzmin, V.A. Rubakov, M. E. Shaposhnikov, Phys. Lett. **155B** (1985) 36.
- [6] P. Arnold, L. McLerran, Phys. Rev. D **36** (1987) 581.
- [7] S. Yu. Khlebnikov, M. E. Shaposhnikov, Nucl. Phys. **B308** (1988) 885.
- [8] V.A. Rubakov, M. E. Shaposhnikov, Usp. Fis. Nauk **166** (1996) 493, hep-ph/9603208.
- [9] A. D. Linde, Phys. Lett. **96B (1980)** 289.
- [10] D.Yu. Grigoriev and V.A. Rubakov, Nucl.Phys. **B299** (1988) 67.
- [11] J. Ambjørn et al., Nucl. Phys. **B253** (1991) 346; J. Ambjørn and K. Farakos, Phys. Lett. **B294** (1992) 248.
- [12] J. Ambjørn, A. Krasnitz, Phys. Lett. **B362** (1995) 97.
- [13] G. D. Moore, hep-ph/9603384.
- [14] G. D. Moore, hep-lat/9605001.
- [15] W. H. Tang, J. Smit, hep-lat/9605016.
- [16] G. D. Moore, N. Turok, hep-ph/9608350.
- [17] P. Ginsparg, Nucl. Phys. **B170** (1980) 388.
- [18] T. Appelquist, R. Pisarski, Phys. Rev. D **23** (1981) 2305.

- [19] S. Nadkarni, Phys. Rev. D **27** (1983) 917.
- [20] N.P. Landsman, Nucl. Phys. **B322** (1989) 498.
- [21] K. Kajantie, M. Laine, K. Rummukainen, M. Shaposhnikov, Nucl. Phys. **B458** (1996) 90.
- [22] D. Bödeker, L. McLerran, A. Smilga, Phys. Rev. D **52** (1995) 4675.
- [23] E. Braaten and R. Pisarski, Nucl. Phys. **B337** (1990) 569; Phys. Rev. D **45** (1992) 1827; J. Frenkel and J.C. Taylor, Nucl. Phys. **B334** (1990) 199; J.C. Taylor and S.M.H. Wong, Nucl. Phys. **B346** (1990) 115; R. Efraty, V.P. Nair, Phys.Rev.D **47** (1993) 5601; J.P. Blaizot, E. Iancu, Nucl.Phys. **B417** (1994) 608; V.P. Nair, Phys.Rev.D **50** (1994) 4201.
- [24] E. Wigner, Phys. Rev. **40** (1932) 749; G. E. Uhlenbeck, L. Gropper, Phys. Rev. **41** (1932) 79; J. G. Kirkwood, Phys. Rev. **44** (1933) 31.
- [25] E. M. Lifshitz and L.P. Pitaevsky, Statistical Physics (Pergamon Press, Oxford, 1981).
- [26] J. I. Kapusta, Finite-temperature field theory (Cambridge University Press, 1989).

FIGURES



FIG. 1. Feynman diagrams contributing to mass renormalization.